Returns to education, indeterminacy, and multiple balanced growth paths

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Abstract

We investigate the existence, multiplicity, and indeterminacy of balanced growth paths in an extended Lucas model incorporating physical capital inputs, human capital externalities, and decreasing returns to scale in education. With physical capital in education and increasing social returns in production, social returns to scale in education should be decreasing for the existence of balanced growth; indeterminacy can arise for weaker human capital externalities; and multiple balanced growth paths may emerge with perhaps distinctive dynamic properties: The high-growth steady state may be indeterminate, while the low-growth steady state may be determinate, but not vice versa.

Keywords : Human capital externalities; Determinacy/Indeterminacy; Multiple steady states; Endogenous growth; Returns to scale

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1 Introduction

Diverse development experiences across nations and over time have received a great deal of attention in the literature on economic growth. For example, Lucas (1993) asks why Philippine and South Korea, two countries with very similar starting points in 1960, have had very different growth rates since. At the same time, South Korea has had larger growth volatility than Philippine (World Development Indicators). After two decades, one may now ask why some countries like South Korea or Japan, once known as growth miracles, have had mediocre or sluggish growth more recently,and why some countries like Singapore have forged ahead. One may also ask why the emerging economies have switched to a rapid growth path and sustained the momentum in recent decades. One of the frameworks in which economists try to explain diverse growth experiences is the Uzawa-Lucas model.

The dynamics at steady states, or interchangeably balanced growth paths (BGPs), have long been studied in various versions or extensions of the Uzawa (1965) model with constant returns to scale in production for goods and in education for human capital accumulation. Among them, Bond et al. (1996) find the existence, uniqueness, and saddle-path stability (determinacy[1]) of the BGP, with an extension of the Uzawa model to include physical capital in the education technology, as in Mulligan and Sala-i-Martin (1993), Stokey and Rebelo (1995), and Azariadis et al. (2013). Incorporating positive sector-specific externalities of both physical and human capital in two sectors, Mino (2001) shows that indeterminacy could emerge at a unique steady state even in cases with decreasing private returns to scale and constant social returns to scale. Ladrón-de-Guevara et al. (1997, 1999) find multiple steady states with endogenous leisure. Introducing leisure externalities, however, Azariadis et al. (2013) find a unique BGP.

To explore the mechanics of development, Lucas (1988) incorporates empirically plausible spillovers of average human capital that generate increasing social returns in production in the Uzawa model.[2] The increasing returns to scale via human capital externalities give rise to rich

[1] To avoid the confusion with multiple BGPs, (local) determinacy or indeterminacy always refers to the transition paths converging to a single BGP in this paper.

[2] There is indeed supporting empirical evidence for such spillovers in the literature (e.g. Young, 1928; Basu and Fernald, 1997; Harris and Lau, 1998; Moretti, 2004a, 2004b).

dynamics in endogenous growth models: Benhabib and Perli (1994) find a unique BGP in the Lucas model and multiple BGPs when incorporating endogenous labor. Also, strong enough human capital externalities in excess of the role of physical capital in production ($\gamma > \beta$) and a high enough elasticity of intertemporal substitution (a small σ) are sufficient for indeterminacy or a local continuum of equilibrium paths at the BGP, yet a labor-leisure trade off relaxes the restriction on σ for indeterminacy. Setting an inverse relation between the intertemporal elasticity of substitution and the share parameter of capital, Xie (1994) finds a global continuum of equilibrium paths converging to a unique BGP under the same condition $\gamma > \beta$. While an indeterminate BGP or a continuum of transitional equilibrium paths helps to explain diverse growth experiences in finite time, multiple BGPs help to explain those in the long run.

However, it is questionable whether human capital spillovers driving increasing returns in production are indeed more important than physical capital in production as required for indeterminacy in the Lucas model ($\gamma > \beta$). Also, the Lucas-style models typically use effective labor as the sole input in education with constant returns to scale. Yet, little attention has been paid to such empirically plausible features of education as the physical capital input, returns to scale, and human capital externalities in the Lucas-type models with increasing returns in production via human capital spillovers. According to Bowen (1987) and Jones and Zimmer (2001), physical investment plays a significant role in the education sector. Borjas (1992, 1995), among others, finds empirical evidence for human capital externalities in education. Moreover, Psacharopoulos (1994) and Trostel (2004) present empirical evidence for significantly decreasing private and/or social returns to scale, at least at high levels of education. Usually known as a force for convergence and against sustainable growth, the decreasing returns to scale in education cast doubt about the existence, indeterminacy and multiplicity of a sustainable balanced growth path. Overall, such important considerations in education should affect the dynamic properties of the equilibrium paths as far as the existence, indeterminacy and multiplicity of the BGP are concerned.

The present paper tries to fill in these gaps. We investigate the existence, indeterminacy, and multiplicity of BGPs in an extended Lucas model by incorporating several factors in the education sector: physical capital inputs, human capital externalities, and decreasing returns to scale. In doing so, we do not start with any strong restrictions on factor intensities or externalities for our model. As in Mulligan and Sala-i-Martin (1993), we begin with relatively general forms of technologies and identify the restrictions on the parameters for the existence of balanced growth, viewing the Uzawa (1965) and the Lucas (1988) models as special cases.

The present model makes several contributions. First, with physical capital inputs in education and increasing social returns in production, social returns to scale in education should be decreasing for the existence of BGPs; indeterminate BGPs can arise for weaker human capital externalities and the importance of human capital in education plays a great role in the possibility of indeterminacy; and multiple BGPs may emerge with perhaps distinctive dynamic properties: The high-growth BGP may be indeterminate and the low-growth BGP may be determinate but not vice versa. In particular, our use of the empirically plausible decreasing returns to scale in education strengthens the argument for indeterminacy and multiplicity of BGPs in this type of model. The results help to explain why some countries could achieve extraordinary growth for a few decades and why it is difficult to avoid eventual growth slowdown as experienced in Japan.

The intuition for indeterminacy comes from human capital externalities for increasing returns to scale in production as in the literature (e.g. Benhabib and Perli, 1994; Xie, 1994), and from the more general education technologies in the present model. Starting from any equilibrium path, another one may be justified by saving more and allocating more resources into the education sector, so long as the rate of return of capital increases sufficiently and as consumers have strong enough willingness for intertemporal substitution. Stronger increasing returns in production via human capital spillovers allow the rate of return of capital to increase more. Also, a higher educational output elasticity of human capital in the present model enhances the effectiveness of this intersectoral re-allocation. When physical investment plays a role in the education sector in the present model, the complementarity between physical and human capital promotes the effectiveness of this intersectoral re-allocation further, by allocating more physical capital into the education sector together with human capital. So indeterminacy in the present model could occur for weaker human capital externalities than those in the literature mentioned above, thereby easing the concern about whether the required strength of human capital externalities for indeterminacy is empirically plausible.

The source for multiple BGPs in the present paper hinges on the balance between decreasing private/social returns to scale in education and increasing social returns in production, both via human capital externalities, given strong enough intertemporal substitution, rather than on leisure in the literature. The decreasing return to scale in education tends to induce smaller fractions of available resources for education, while the increasing returns in production via human capital externalities tends to induce the opposite through equilibrium feedback effects. Consequently, the low (high) growth BGP accompanies greater (smaller) shares of human and physical capital used for production. Therefore, an economy at the low-growth BGP avoids some loss from decreasing returns to scale in education but benefits less from increasing returns to scale in production from human capital externalities than at the high-growth BGP. Given that increasing returns to scale in production via human capital externalities are the source for indeterminate BGPs, it is possible to have a pair of a determinate low-growth BGP and an indeterminate high-growth BGP. Whereas it is impossible to have a pair of an indeterminate low-growth BGP and a determinate high-growth BGP. Absent these additional factors in the education sector, the BGP would be unique as in the literature mentioned above.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 analyzes the equilibrium paths, and the existence and multiplicity of BGPs. Section 4 analyzes the local determinacy/indeterminacy of the BGPs. The last section concludes the paper.

2 The model

The model extends that in Lucas (1988) to incorporate the physical capital input, human capital externalities, and decreasing returns to scale in the education sector. Starting with initial stocks of human and physical capital $H(0)$ and $K(0)$, the representative agent maximizes his utility derived from consumption $C(t)$ over an infinite horizon by the choice of the fractions of human and physical capital $(u(t), \nu(t))$ and the path of consumption:

$$\max_{C(t),u(t),\nu(t)} \int_0^\infty \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \tag{1}$$

subject to the technologies for production and education, and the budget constraint (all time subscripts omitted):

$$Y = A(\nu K)^\beta (uH)^{1-\beta} H_a^\gamma, \tag{2}$$

$$\dot{H} = B[(1-\nu)K]^\alpha [(1-u)H]^\eta H_a^{b(\gamma)} \equiv X, \tag{3}$$

$$Y = C + \dot{K}, \tag{4}$$

taking the average human capital H_a as given.

Here, $1/\sigma > 0$ is the elasticity of intertemporal substitution, $\rho > 0$ is the rate of time preference, $\beta \in [0,1]$ and $\alpha \in [0,1]$ are the output elasticities of physical capital in production and in education respectively, $\eta \in [0,1]$ is the output elasticity of human capital in education, and $\gamma \geq 0$ and $b(\gamma)$ are the degrees of human capital externalities in production and education respectively. The exact form of $b(\gamma)$ will be pinned down in Section 3.1, where we discuss the existence of balanced growth.

Some plausible assumptions on the parameters are given as follows. First, decreasing returns to scale in the education sector are allowed according to the aforementioned empirical evidence:

Assumption 1: $\boldsymbol{\alpha + \eta \leq 1}$.

Second, physical capital plays a more important role in production than in education:

Assumption 2: $\boldsymbol{0 \leq \alpha < \beta}$.

Third, production is typically regarded as more physical capital intensive than education. Namely, the contribution of human capital, relative to the contribution of physical capital, is larger in education than in production:

Assumption 3: $\boldsymbol{\eta > \alpha \frac{1-\beta}{\beta}}$ **(or** $\boldsymbol{\frac{\eta}{\alpha} > \frac{1-\beta}{\beta}}$**)**.

3 The equilibrium and balanced growth paths

The optimization problem in (1)-(4) is formulated in the current-value Hamiltonian:

$$\mathcal{H} = \frac{C^{1-\sigma}-1}{1-\sigma} + \mu\left[A(\nu K)^{\beta}(uH)^{1-\beta}H_a^{\gamma} - C\right] + \lambda B[(1-\nu)K]^{\alpha}[(1-u)H]^{\eta}H_a^{b(\gamma)},$$

where μ and λ are the Lagrangian multipliers. The first-order conditions are:

$$C: \; C^{-\sigma} - \mu = 0, \tag{5}$$

$$K: \; \mu\beta Y/K + \lambda\alpha X/K = \rho\mu - \dot{\mu}, \tag{6}$$

$$H: \; \mu(1-\beta)Y/H + \lambda\eta X/H = \rho\lambda - \dot{\lambda}, \tag{7}$$

$$\nu: \; \mu\beta Y/\nu - \lambda\alpha X/(1-\nu) = 0, \tag{8}$$

$$u: \; \mu(1-\beta)Y/u - \lambda\eta X/(1-u) = 0, \tag{9}$$

and the transition equations concerning the state variables in the Hamiltonian.

The transversality conditions are:

$$\lim_{t\to\infty} \mu e^{-\rho t}K = 0,$$

$$\lim_{t\to\infty} \lambda e^{-\rho t}H = 0.$$

The representative agent treats H_a as exogenous. In equilibrium, however, $H_a = H$.

The first-order conditions can be simplified into an autonomous system of differential equations concerning the control and state variables as follows. First, equations (8) and (9) imply the expressions for λ/μ and μ/λ:

$$\frac{\lambda}{\mu} = \frac{1-\nu}{\nu}\frac{\beta Y}{\alpha X}, \tag{10}$$

$$\frac{\mu}{\lambda} = \frac{u}{1-u}\frac{\eta X}{(1-\beta)Y}. \tag{11}$$

Substituting them into (6) and (7) yields growth rates of the multipliers:

$$\frac{\dot{\mu}}{\mu} = \rho - \frac{1}{\nu}\beta Y/K, \tag{12}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - \frac{1}{1-u}\eta X/H. \tag{13}$$

From (5) and (12), the growth rate of consumption is:

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left(\frac{\beta}{\nu}\frac{Y}{K} - \rho\right) = \frac{1}{\sigma}\left[\beta A\left(\frac{uH}{\nu K}\right)^{1-\beta} H^{\gamma} - \rho\right]. \tag{14}$$

From (2) and (4), the growth rate of physical capital is:

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K} = A\nu\left(\frac{uH}{\nu K}\right)^{1-\beta} H^{\gamma} - \frac{C}{K}. \tag{15}$$

From (3), the growth rate of human capital is:

$$\frac{\dot{H}}{H} \equiv \frac{X}{H} = B[(1-\nu)K]^{\alpha}[(1-u)H]^{\eta}H^{b-1}. \tag{16}$$

The derivation of the growth rate of u, the fraction of human capital used in production, takes several steps. First, differentiating (8) with respect to time yields:

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{Y}}{Y} - \frac{\dot{\nu}}{\nu} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{X}}{X} + \frac{\dot{\nu}}{1-\nu}. \tag{17}$$

Next, multiplying equations (10) with (11) on both sides gives

$$\frac{\eta\beta}{\alpha(1-\beta)}\frac{1-v}{\nu}\frac{u}{1-u} = 1, \tag{18}$$

which implies

$$\nu = \frac{u}{D + (1-D)u}, \tag{19}$$

$$D \equiv \frac{\alpha(1-\beta)}{\eta\beta}.$$

From Assumption 3, $0 \leq D < 1$, which will be used frequently later. The unique and positive relationship between ν and u in (19) comes from the complementarity between physical and human capital in production and in education. Notice that ν equals one when $\alpha = 0$, which is the case in the original Lucas model. Differentiating (18) with respect to time leads to

$$\frac{\dot{u}}{u} - \frac{\dot{\nu}}{\nu} = \frac{\dot{\nu}}{1-\nu} - \frac{\dot{u}}{1-u}. \tag{20}$$

Finally, using (2), (3), (12), (13) and (20) in (17) for substitution gives rise to

$$\begin{aligned} \frac{\dot{u}}{1-u} &= \left[\frac{\beta-\alpha}{D+(1-D)u} + 1 - \eta - \beta\right]^{-1} \left[(\beta-\alpha)\left(\frac{Y}{K} - \frac{C}{K}\right)\right. \\ &\qquad \left. +(1-\eta-b-\beta+\gamma)\frac{X}{H} - \frac{1}{\nu}\frac{\beta Y}{K} + \frac{1}{1-u}\frac{\eta X}{H}\right], \end{aligned} \tag{21}$$

where the first factor on the right-hand side is positive as shown below:

$$\begin{aligned} Q &\equiv \frac{\beta-\alpha}{D+(1-D)u} + 1 - \eta - \beta \\ &= \frac{\beta-\alpha+(1-\eta-\beta)[D+(1-D)u]}{D+(1-D)u} \\ &= \frac{\beta(1-D)(1-u) - \alpha[D+(1-D)u+1-D-(1-D)u] + (1-\eta)[D+(1-D)u]}{D+(1-D)u} \\ &= \frac{(\beta-\alpha)(1-D)(1-u) + (1-\eta-\alpha)[D+(1-D)u]}{D+(1-D)u} > 0. \end{aligned}$$

This factor will be used repeatedly.

The equilibrium paths of variables (C, H, K, u, ν) are determined by (14), (15), (16), (19) and (21). We denote the right-hand sides of (14), (15), (16) and (21), for the growth rates of consumption C, physical capital K, human capital H, and the fraction of human capital used in production u, as $\tilde{\Gamma}(K, H, u)$, $\tilde{\Delta}(C, K, H, u)$, $\tilde{\Theta}(K, H, u)$ and $\tilde{\Omega}(C, K, H, u)$ respectively. Then, the equilibrium path can be written in a block of four differential equations:

$$\dot{C} = C\tilde{\Gamma}(K, H, u) \tag{22}$$

$$\dot{K} = K\tilde{\Delta}(C, K, H, u) \tag{23}$$

$$\dot{H} = H\tilde{\Theta}(K, H, u) \tag{24}$$

$$\dot{u} = (1-u)\tilde{\Omega}(C, K, H, u). \tag{25}$$

This autonomous system of differential equations allows us to determine the BGP more conveniently.

3.1 *The existence of balanced growth paths*

A balanced or steady state growth path (BGP) refers to the stage of an equilibrium path on which the growth rates of Y, C, H, K and the fractions of human and physical capital used in production (u and ν) become constant over time. From (15), human capital cannot share the same growth rate with physical capital in the long run for $\gamma > 0$. On the BGP, output Y, physical capital K, and consumption C all grow at the same constant rate, denoted by g^*. However, human capital H grows at the rate $(1-\beta)g^*/(1-\beta+\gamma)$, as in the Lucas model. We apply this result into (16) and pin down the specific form of $b(\gamma)$ linking the externality in the education sector to the externality in the production sector for the existence of BGPs:

$$b(\gamma) = 1 - \eta - \alpha\frac{1-\beta+\gamma}{1-\beta}. \tag{26}$$

This yields non-increasing social returns to scale in the education sector, as $\alpha + \eta + b(\gamma) = 1 - \alpha\gamma/(1-\beta) \leq 1$. Intuitively, should both sectors demonstrate increasing social returns to scale, the agent's optimization problem in (1)-(4) would be confronted with explosive growth and thus undermine the existence of BGPs.

As we start from a quite general form of production functions in both sectors, a wide range of parameter possibilities are permitted, including both the Uzawa and the Lucas model as special cases. The Uzawa model is the case in which $\gamma = 0$, $\alpha = 0$, $\eta = 1$ and $b(\gamma) = 0$, while the Lucas model is the case in which $\gamma > 0$, $\alpha = 0$, $\eta = 1$ and $b(\gamma) = 0$. From (26), the sign of human capital externalities in education may be positive or negative, depending on the private returns to scale in education. For instance, $b(\gamma)$ is positive as long as the private returns to scale are sufficiently decreasing so that $1-\eta-\alpha(1-\beta+\gamma)/(1-\beta) > 0$. If both sectors demonstrate privately constant returns to scale ($\eta = 1-\alpha$), then $b(\gamma) = -\alpha\gamma/(1-\beta)$ is non-positive and our model belongs to the class of models described in Mulligan and Sala-i-Martin (1993) and satisfies their necessary condition for endogenous growth. In fact, private and social returns to scale in education could be both decreasing as found empirically in the literature mentioned earlier.

3.2 *The multiplicity of balanced growth paths*

For analytical convenience of the dynamic system, we now simplify the system in (22)-(25) by reducing one dimension. Let $z \equiv Z/H^{1+\frac{\gamma}{1-\beta}}$ be the human-capital-adjusted value of the variable Z, where $Z = Y$, K, or C. The system can be transformed as

$$\dot{k} = A\nu^{\beta}u^{1-\beta}k^{\beta} - c - \frac{1-\beta+\gamma}{1-\beta}B(1-\nu)^{\alpha}(1-u)^{\eta}k^{1+\alpha}, \tag{27}$$

$$\dot{c} = c\left[\frac{1}{\sigma}\left(\beta A\nu^{\beta-1}u^{1-\beta}k^{\beta-1} - \rho\right) - \frac{1-\beta+\gamma}{1-\beta}B(1-\nu)^{\alpha}(1-u)^{\eta}k^{\alpha}\right], \tag{28}$$

$$\begin{aligned}\dot{u} = {}& (1-u)Q^{-1}\left\{[(\beta-\alpha)\nu-\beta]A\left(\frac{u}{\nu k}\right)^{1-\beta} + (\alpha-\beta)\frac{c}{k}\right. \\ & \left. + \left[1-\eta-b-\beta+\gamma+\frac{\eta}{1-u}\right]B(1-\nu)^{\alpha}(1-u)^{\eta}k^{\alpha}\right\},\end{aligned} \tag{29}$$

where ν and u have a one-for-one positive relationship in (19). On the BGP where $\dot{k} = \dot{c} = \dot{u} = 0$, one can use (27)-(29) to solve for c^*, k^*, u^* and ν^*.

From (16) and (26), the balanced growth rate g can be expressed in terms of (k, u, v):

$$g \equiv \frac{\dot{Y}}{Y} = \frac{1-\beta+\gamma}{1-\beta}\frac{\dot{H}}{H} = \frac{1-\beta+\gamma}{1-\beta}B(1-\nu)^{\alpha}(1-u)^{\eta}k^{\alpha}.$$

From this growth equation, (19), and (27)-(29), the growth rate on the BGP, g^*, is determined implicitly by

$$\begin{aligned}g^{1-\eta-\alpha} = {}& \frac{1-\beta+\gamma}{1-\beta}B\left[\frac{\alpha(1-\beta)}{\eta\beta}\right]^{\alpha}\left[\frac{\eta(1-\beta)}{-\gamma g+(1-\beta+\gamma)(\sigma g+\rho)}\right]^{\eta+\alpha} \times \\ & \left(\frac{\beta A}{\sigma g+\rho}\right)^{\frac{\alpha}{1-\beta}}.\end{aligned} \tag{30}$$

Here, the balanced growth rate is shared by physical capital K, output Y, and consumption C, while human capital H grows at a lower rate $(1-\beta)g^*/(1-\beta+\gamma)$ as long as $\gamma > 0$. Substituting g^* into (19) and (27)-(29), we solve the steady state values of the other variables as functions of g^*:

$$u^* = 1 - \frac{\eta(1-\beta)g^*}{-\gamma g^* + (1-\beta+\gamma)(\sigma g^* + \rho)}, \tag{31}$$

$$\nu^* = 1 - \frac{(1-\beta)\eta D g^*}{-[\gamma + (1-\beta)\eta(1-D)]g^* + (1-\beta+\gamma)(\sigma g^* + \rho)}, \tag{32}$$

$$k^* = \left(\frac{\beta A}{\sigma g^* + \rho}\right)^{\frac{1}{1-\beta}} \frac{u^*}{\nu^*}, \tag{33}$$

$$c^* = \frac{\nu^* k^*}{\beta}(\sigma g^* + \rho) - k^* g^*. \tag{34}$$

Moreover, the transversality condition requires the balanced growth rate to satisfy:

$$\sigma g^* + \rho > g^*.$$

It can be verified that this constraint is sufficient to ensure that the solution is interior, i.e. $C,\ K,\ H > 0$ and $0 < \nu,\ u < 1$. It is now ready to explore the conditions for a unique BGP or multiple BGPs.

Proposition 1. *The BGP is unique if $\sigma > \gamma/(1-\beta+\gamma)$ or if $\eta = 1$. Otherwise, multiple BGPs are possible and constructed for $\sigma \leq \gamma/(1-\beta+\gamma)$ and $\eta < 1$. For the case that private returns to scale are constant in education ($\alpha + \eta = 1$), there are multiple BGPs if and only if conditions (35)-(37) are met.*

Proof. The proof is based on (30) that determines the balanced growth rate. The left-hand side of (30) is strictly increasing and concave in g if $0 < \alpha + \eta < 1$; otherwise, it equals one if $\alpha + \eta = 1$. Whereas the right-hand side, denoted as $R(g)$, is positive under the transversality condition $(\sigma - 1)g + \rho > 0$. Then,

$$R'(g) = -(\alpha+\eta)R(g)\left[\frac{-\gamma + \sigma(1-\beta+\gamma)}{-\gamma g + (1-\beta+\gamma)(\sigma g + \rho)}\right] - R(g)\frac{\alpha\sigma}{(1-\beta)(\sigma g + \rho)},$$

which is negative for $\sigma > \gamma/(1-\beta+\gamma)$ and may be positive or negative otherwise; and

$$R''(g) = R(g)\left\{\frac{(\alpha+\eta)[-\gamma+\sigma(1-\beta+\gamma)]^2}{[-\gamma g+(1-\beta+\gamma)(\sigma g+\rho)]^2} + \frac{\alpha\sigma^2}{(1-\beta)(\sigma g+\rho)^2}\right\} +$$
$$R(g)\left\{\frac{(\alpha+\eta)[-\gamma+\sigma(1-\beta+\gamma)]}{[-\gamma g+(1-\beta+\gamma)(\sigma g+\rho)]} + \frac{\alpha\sigma}{(1-\beta)(\sigma g+\rho)}\right\}^2 > 0.$$

The balanced growth rate g^* is unique when $\sigma > \gamma/(1-\beta+\gamma)$, since then the right (left) hand side of equation (30) is strictly decreasing (non-decreasing) with respect to g. Moreover, if $\eta = 1$, and hence $\alpha = 0$, as in Lucas (1988), Benhabib and Perli (1994), and Xie (1994), a unique reduced-form solution arises from (30):

$$g = \frac{(1-\beta+\gamma)(B-\rho)}{\sigma(1-\beta+\gamma)-\gamma}.$$

From (26) , $b(\gamma) = 0$ (no externalities in education) in this case. Note that if $\gamma \to 0$ then $g \to (B-\rho)/\sigma$.

If $\sigma \leq \gamma/(1-\beta+\gamma)$ and $0 < \eta < 1$, there are several cases in which multiple BGPs may arise. In the first case with $\alpha = 0$ and $0 < \eta < 1$ (decreasing private returns to education), the left-hand side of (30) is increasing and concave, while the right-hand side is increasing and convex in g because now $R'(g) > 0$ for $\alpha = 0$ and $\sigma \leq \gamma/(1-\beta+\gamma)$, and $R''(g) > 0$. Also, the left-hand side $g^{1-\eta}$ starts at the value zero and rises at an infinite rate $(1-\eta)g^{-\eta}$ at $g = 0$ but eventually becomes flat at higher values of g. Whereas, the right-hand side $R(g)$ starts above zero at $g = 0$ and rises at an increasing rate. So it is very likely to have multiple BGPs. See Figure 1 for an example, based on the parametrization below the figure. Both the balanced growth rates in this figure meet the transversality condition.

In the second case with $\alpha > 0$ and $\alpha + \eta = 1$, the left-hand side of (30) equals one, while the right-hand side may first decline at low growth rates, since now the second term of $R'(g)$ is negative, but may eventually increase at high growth rates as in the first case, as shown in Figure 2. As $R''(g) > 0$, the necessary and sufficient conditions for multiple roots to (30) in this case are:

(a). $R(0) \geq 1$,

(b). $R'(0) < 0$ and

(c). $\min_g \{R(g)\} < 1$.

Condition (a) and (b) require respectively

$$R(0) = B\left[\frac{\alpha(1-\beta)}{\eta\beta}\right]^{\alpha}\frac{\eta}{\rho}\left(\frac{\beta A}{\rho}\right)^{\frac{\alpha}{1-\beta}} \geq 1. \tag{35}$$

$$\sigma > \frac{\gamma(1-\beta)}{(1+\alpha-\beta)(1-\beta+\gamma)}. \tag{36}$$

The lower bound (on σ) for $R'(0) < 0$ is smaller than the upper bound, $\sigma < \gamma/(1-\beta+\gamma)$, for possible multiple BGPs. Last, $R'(g)$ switches its sign from negative to positive at $\underline{g}$ such that $R'(\underline{g}) = 0$ where

$$\underline{g} = \frac{\rho\alpha\sigma(1-\beta+\gamma) - \rho(1-\beta)[\gamma - \sigma(1-\beta+\gamma)]}{\sigma(1-\beta+\alpha)[\gamma-\sigma(1-\beta+\gamma)]}.$$

So condition (c) requires

$$R(\underline{g}) = \frac{1-\beta+\gamma}{1-\beta}B\left[\frac{\alpha(1-\beta)}{\eta\beta}\right]^{\alpha}(\beta A)^{\frac{\alpha}{1-\beta}}\frac{\eta\alpha\sigma}{\gamma-\sigma(1-\beta+\gamma)}\left(\frac{1}{\sigma\underline{g}+\rho}\right)^{\frac{1-\beta+\alpha}{1-\beta}} < 1. \tag{37}$$

In the last case with $\alpha > 0$ and $\alpha + \eta < 1$, the left-hand side of (30) is increasing and concave in the growth rate, while the right-hand side is (eventually) an increasing and convex function because $R'(g) > 0$ for high enough values of g and $R''(g) > 0$. So multiple BGPs are likely to arise, as shown in Figure 3, where transversality holds as well. $\square$

The BGP is unique in the Uzawa (Lucas) model with (without) physical capital in education and with constant private and social returns to scale in education. From the extensions of the education technology, the present model makes a contribution to produce multiple BGPs through decreasing returns to scale in education. The higher balanced growth rate is associated with higher fractions of human and physical capital for education (low u^* and low v^*) from (31) and (32).

The reason for multiple BGPs hinges on the balance between decreasing private or social returns in education and increasing social returns in production, given strong enough intertemporal elasticities of substitution. The former (latter) tends to reduce (increase) the fractions of various available resources for education. The implication for diverse growth experiences complements that from the indeterminacy of a unique BGP in the literature. While the indeterminacy of a unique BGP permits various equilibrium paths in transition, multiple BGPs justify very different growth rates across similar countries (such as Korea and Philippine) not only in the transition but also in the long run. They also justify very different growth rates across long periods of time for the same country (like Japan) in the absence of fundamental structural or institutional changes.

There is no consensus in the literature on the value of the intertemporal elasticity of substitution, a critical parameter for uniqueness versus multiplicity of BGPs (and determinacy versus indeterminacy later). While small intertemporal elasticities of substitution ($\sigma \geq 1$) are typically used in the business cycle literature, there are empirical findings supporting elastic intertemporal substitution in the range of $\sigma = 0.5$ to $\sigma = 1$. Such estimates are based on models with human capital and education components in Keane and Wolpin (2001) and Imai and Keane (2004), with saving and financial market behaviors in Mulligan (2002) and Vissing-Jorgensen and Orazio (2003), and with variations in the capital income tax rates in Gruber (2006). Some of these estimates are in an intergenerational framework. Notice that the Lucas model can also be interpreted as an intergenerational model. We now move on to analyze stability properties of BGPs.

4 Stability properties of balanced growth paths

To study the stability property of a BGP, we first calculate the Jacobian matrix of the dynamic system in (27)-(29) on the BGP:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \tag{38}$$

where

$$
\begin{aligned}
J_{11} &= \frac{\partial \dot{k}}{\partial k} = -(1+\alpha)g^* + \nu^*(\sigma g^* + \rho), \\
J_{12} &= \frac{\partial \dot{k}}{\partial c} = -1, \\
J_{13} &= \frac{\partial \dot{k}}{\partial u} = \left(\eta + \frac{\alpha\nu^*}{u^*}\right)\frac{k^*}{1-u^*}g^* + \left(\frac{1-\beta}{\beta} + \frac{D\nu^*}{u^*}\right)\frac{\nu^*}{u^*}k^*(\sigma g^* + \rho), \\
J_{21} &= \frac{\partial \dot{c}}{\partial k} = -\frac{c^*}{k^*}\left[\alpha g^* + \frac{1-\beta}{\sigma}(\sigma g^* + \rho)\right], \\
J_{22} &= \frac{\partial \dot{c}}{\partial c} = 0, \\
J_{23} &= \frac{\partial \dot{c}}{\partial u} = \frac{c^*}{u^*(1-u^*)}\left[(\alpha\nu^* + \eta u^*)g^* + \frac{1-\beta}{\sigma}(\nu^* - u^*)(\sigma g^* + \rho)\right], \\
J_{31} &= \frac{\partial \dot{u}}{\partial k} = (1-u^*)Q^{-1}\frac{1}{k^*}\left\{\alpha\frac{1-\beta}{1-\beta+\gamma}\left(\alpha + \frac{\alpha\gamma}{1-\beta} - \beta + \gamma\right.\right. \\
&\quad \left.\left. + \frac{\eta}{1-u^*}\right)g^* + (\alpha-\beta)g^* + [(\beta-\alpha)\nu^* + 1 - \beta](\sigma g^* + \rho)\right\}, \\
J_{32} &= \frac{\partial \dot{u}}{\partial c} = -(1-u^*)Q^{-1}\frac{\beta-\alpha}{k^*}, \\
J_{33} &= \frac{\partial \dot{u}}{\partial u} = (1-u^*)Q^{-1}\left\{\left[\left(\beta - \frac{\alpha\gamma}{1-\beta} - \alpha - \gamma\right)\frac{1-\beta}{1-\beta+\gamma}\frac{1}{1-u^*}\left(\eta + \frac{\alpha\nu^*}{u^*}\right)\right.\right. \\
&\quad \left. + \frac{\eta}{(1-u^*)^2}\frac{1-\beta}{1-\beta+\gamma}\left(1 - \frac{\alpha\nu^*}{u^*} - \eta\right)\right]g^* \\
&\quad \left. + \frac{(\beta-\alpha)\nu^*[(1-\beta)u^* + \beta D\nu^*] + \beta(1-\beta)(D\nu^* - u^*)}{\beta u^{*2}}(\sigma g^* + \rho)\right\},
\end{aligned}
$$

in which $D \in [0, 1)$ is given below (19) and $Q > 0$ is given below (21).

Adopting the conventional approach in the literature, we identify the signs of the eigenvalues of the Jacobian matrix by calculating several characteristics: the determinant, the trace, and the function $\mathrm{B}(J)$ which will be defined later. The **determinant** of the Jacobian matrix is

$$
\det(J) = J_{13}J_{21}J_{32} - J_{23}J_{31} + J_{21}J_{33} - J_{11}J_{23}J_{32}. \tag{39}
$$

We define $\theta \equiv (\rho,\ \sigma,\ A,\ B,\ \gamma,\ \beta,\ \eta,\ \alpha,\ g^*)$ and $\theta \in \Theta$, where $\Theta \subset R^5_{++}\times(0,\ 1)^2\times[0,\ 1)\times R_{++}$

such that Assumptions 1-3 are all met and equation (30) is satisfied. We have to include g^* here, as we have already shown that it is not in general uniquely determined by the parameters and multiple BGPs under the same parametrization may not share the same stability feature.

This space Θ can be separated into three mutually-exclusive subsets:

$$\begin{aligned}\Theta_1 \;&\equiv\; \Bigg\{\theta\in\Theta\,|\,Z\equiv-\left(\frac{1-\alpha-\eta}{\sigma}\right)(1-\beta+\gamma)\left(\frac{\sigma g^*+\rho}{g^*}\right)^2+\\ &\quad\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right)\right]\frac{\sigma g^*+\rho}{g^*}+\frac{\alpha\gamma}{1-\beta}<0\Bigg\},\\ \Theta_2 \;&\equiv\; \{\theta\in\Theta\,|\,Z>0\} \text{ and}\\ \Theta_3 \;&\equiv\; \{\theta\in\Theta\,|\,Z=0\},\end{aligned}$$

and the determinant can be signed in these subsets as shown in the Appendix:

Lemma 1. *The sign of the determinant of the Jacobian matrix is given by*

(i) $\det(J)<0$ *if* $\theta\in\Theta_1$,

(ii) $\det(J)>0$ *if* $\theta\in\Theta_2$,

(iii) $\det(J)=0$ *if* $\theta\in\Theta_3$.

Hereafter, we will focus on the region $\Theta_1\bigcup\Theta_2$ only. The **trace** of the Jacobian matrix is

$$\mathrm{tr}(J) \;=\; J_{11}+J_{33}. \tag{40}$$

Moreover, $\mathbf{B}(\boldsymbol{J})$ is defined as that in Benhabib and Perli (1994):

$$\begin{aligned}\mathrm{B}(J) \;&=\; \begin{vmatrix}J_{11} & -1\\ J_{21} & 0\end{vmatrix}+\begin{vmatrix}0 & J_{23}\\ J_{32} & J_{33}\end{vmatrix}+\begin{vmatrix}J_{11} & J_{13}\\ J_{31} & J_{33}\end{vmatrix}\\ &=\; J_{21}-J_{23}J_{32}+J_{11}J_{33}-J_{13}J_{31}.\end{aligned} \tag{41}$$

We now present the stability of the BGP shown in the Appendix. We will not state the transversality condition $\sigma g^*+\rho>g^*$ explicitly in the following propositions, though it has to hold.

Proposition 2. *The BGP is determinate if and only if* $\theta\in\Theta_1$. *And the following conditions* *are sufficient (but not necessary) for determinacy:*

$$(i) \quad \sigma \geq 1 \quad \textit{or else if}$$

$$(ii) \quad \eta \leq \frac{1-\beta+\alpha\sigma}{(1-\sigma)(1-\beta+\gamma)} - \alpha.$$

The BGP is always determinate if the intertemporal substitution is inelastic enough ($\sigma \geq 1$), which is consistent with the literature. It implies that indeterminacy cannot emerge when σ is relatively large. Intuitively, the less agents are willing to shift resources intertemporally, the smaller the possibility of alternative converging paths to the BGP. Alternatively, if $\sigma < 1$, a low enough output elasticity of human capital in education (η) is sufficient for a determinate BGP. It implies that indeterminacy cannot emerge if η is too low because of decreasing private returns to scale in education, a feature captured in the present model as opposed to the related literature with $\eta = 1$. The intuition comes from the effectiveness of the intersectoral re-allocation of human capital: A low output elasticity of human capital in education hinders this effectiveness, which is necessary for rationalizing an alternative equilibrium path by accelerating/decelerating human capital accumulation. We present the condition for indeterminacy and relegate the proof to the Appendix.

Proposition 3. *There exists a partition of Θ_2 into two subsets Θ_2^I and Θ_2^U, with $\Theta_2^I \bigcap \Theta_2^U = \emptyset$ and $\Theta_2^I \bigcup \Theta_2^U = \Theta_2$, such that the BGP is indeterminate if $\theta \in \Theta_2^I$, and there is no converging path to the BGP (an unstable BGP) if $\theta \in \Theta_2^U$.*

It is difficult to identify the exact constraints for Θ_2^I and Θ_2^U in general, i.e. the necessary and sufficient conditions for indeterminate and unstable BGPs. However, in some special cases, we can find the sufficient conditions for an indeterminate BGP (i.e. a subset of Θ_2^I). In the more general case, we can only use numerical simulations. Thus, we present the results in three cases separately: the case with $\alpha = 0$ and $\eta \leq 1$ to focus on the role of decreasing private returns to scale in education; the case with $\eta = 1 - \alpha$ to focus on the role of physical capital inputs in education; and the case with $\alpha > 0$ and $\eta < 1 - \alpha$.

4.1 ***The case*** $\alpha = 0$ ***and*** $\eta \leq 1$

This case is the closest to the original Lucas model where physical capital does not play any role in the education sector ($\eta = 1$). When $\eta < 1$, the private returns to scale are decreasing in the education sector but the social returns to scale are constant, in the presence of human capital externalities in education under the condition for balanced growth in (26):

$$\alpha + \eta + \underbrace{\left(1 - \eta - \alpha \frac{1 - \beta + \gamma}{1 - \beta}\right)}_{b(\gamma)} = 1 \text{ when } \alpha = 0.$$

Here, the difference between the private and social returns to scale comes from the positive human capital externality in education $b(\gamma) = 1 - \eta$, according to (26). The necessary and sufficient condition for determinacy in this case is given below (see the Appendix):

Proposition 4. *The BGP is determinate if and only if*

$$\eta < 1 - \frac{g^*}{\rho}\left(\frac{\gamma}{1 - \beta + \gamma} - \sigma\right).$$

This is a corollary of Proposition 2. When $\eta = 1$, the present model becomes the same as the original Lucas model analyzed in Benhabib and Peril (1994) for indeterminacy, where the balanced growth rate can be solved analytically from (30). The constraint in the proposition with $\eta = 1$, under the transversality condition $\sigma g^* + \rho > g^*$, generates the same result as Proposition 1 in their paper. As $\eta \leq 1$ in this case, if $\sigma > \gamma/(1 - \beta + \gamma)$, then the right-hand side of the inequality is always greater, which is sufficient for a determinate BGP regardless of the size of η; however, if $\sigma \leq \gamma/(1 - \beta + \gamma)$, then a small enough output elasticity of human capital in education (η) is necessary and sufficient for a determinate BGP. As shown in the Appendix, the sufficient conditions for indeterminacy are:

Proposition 5. *The BGP is indeterminate if*

$$(i) \quad \eta > 1 - \frac{g^*}{\rho}\left(\frac{\gamma}{1-\beta+\gamma} - \sigma\right) \quad \textit{and}$$

$$(ii) \quad \eta \geq 1 - \frac{(\gamma-\beta)(1-\beta)}{(1-\beta+\gamma)\left(\sigma + \frac{\rho}{g^*}\right) - \gamma + (\gamma-\beta)(1-\beta)},$$

where condition (i) is necessary.

Both constraints for indeterminacy require a large enough output elasticity of human capital in the education sector. If $\eta = 1$ (constant private returns to scale in education), then the proposition coincides with Proposition 2(i) in Benhabib and Perli (1994). In contrast to this special situation with $\eta = 1$, the proposition implies that indeterminacy is still possible to emerge when the education sector demonstrates decreasing private returns to scale ($\eta < 1$). For instance, the parametrization $\rho = 0.05$, $\sigma = 0.3$, $A = 1$, $B = 0.046$, $\gamma = 0.5$, $\beta = 0.33$, $g^* = 0.03$ and $\eta = 0.95$ can be shown to generate an indeterminate BGP. However, there are two implicit constraints in the proposition on σ and γ respectively: Condition (i) implies that $\sigma < \gamma/(1-\beta+\gamma)$, and condition (ii) implies that $\gamma \geq \beta$, both because η is no greater than one (Assumption 1). Therefore, relatively elastic intertemporal substitution is necessary for an indeterminate BGP, in line with the discussion under Proposition 4. Moreover, in this parameter region of indeterminacy, a relatively large human capital spillover effect is required: $\gamma > \beta$ as in Benhabib and Perli (1994) and Xie (1994), which may be questionable for empirical plausibility. However, we will see how this constraint loosens in the next case.

4.2 *The case* $\alpha > 0$ *and* $\eta = 1 - \alpha$

This is the case when physical capital plays a role and private returns to scale are constant in the education sector. In this case, the existence of a BGP requires a negative externality of human capital (due to congestion for instance) in the education sector, according to (26):

$$b(\gamma) = 1 - (1-\alpha) - \alpha\frac{1-\beta+\gamma}{1-\beta} = -\frac{\alpha\gamma}{1-\beta} < 0 \text{ when } \alpha > 0.$$

The necessary and sufficient condition for determinacy in this case is derived in the Appendix and given below:

Proposition 6. *The BGP is determinate if and only if*

$$\eta < 1 - \frac{(1-\beta)\left(\sigma + \frac{\rho}{g^*}\right)\left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\right]}{(1-\beta+\gamma)\left(\sigma + \frac{\rho}{g^*}\right) - \gamma}.$$

This is another corollary of Proposition 2. Again, Proposition 1 of Benhabib and Perli (1994) can be replicated as a special case of Proposition 6, if we set $\alpha = 0$ and impose the transversality condition $\sigma g^* + \rho > g^*$ explicitly. Similar to Proposition 4, the BGP is determinate if and only if the output elasticity of human capital in education is small enough. The condition $\sigma > \gamma/(1-\beta+\gamma)$ is also sufficient for determinacy in this case, since then the right-hand side of the inequality in the proposition is greater than one and thus the inequality is always satisfied. The sufficient conditions for indeterminacy are derived in the Appendix and given below:

Proposition 7. *The BGP is indeterminate if*

$$(i) \quad \eta > 1 - \frac{(1-\beta)\left(\sigma + \frac{\rho}{g^*}\right)\left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\right]}{(1-\beta+\gamma)\left(\sigma + \frac{\rho}{g^*}\right) - \gamma} \quad \textit{and}$$

$$(ii) \quad \gamma \geq \frac{1-\beta}{1+\alpha-\beta}(\beta - \alpha),$$

and condition (i) is necessary.

These conditions coincide exactly with those in Proposition 2(i) of Benhabib and Perli (1994), if we let $\alpha = 0$ and impose the transversality condition explicitly. However, compared with their finding, the more generalized result in the present model relaxes the constraint on the strength of human capital externalities in production (γ) for indeterminacy, while an additional constraint on the educational output elasticity of human capital (η) has to be satisfied, after introducing physical input in education ($\alpha > 0$). Similar to Proposition 5, there is one implicit constraint $\sigma < \gamma/(1-\beta+\gamma)$, which can be derived from condition (i), where the right-hand side has to be smaller than one as $\eta \leq 1$. Moreover, $\sigma < \gamma/(1-\beta+\gamma)$ is actually required for an indeterminate BGP, since condition (i) is a necessary condition. This is consistent with the sufficiency of $\sigma > \gamma/(1-\beta+\gamma)$ for determinacy shown under Proposition 6.

As illustrated in Figure 4, region I+II is the necessary region for indeterminacy, constrained by condition (i) alone; while region I is the sufficient region, constrained by conditions (i) and (ii). For region II, the BGP is either indeterminate or unstable (cannot be inferred by Proposition 7). As the reliance on the strength of human capital externalities for indeterminacy declines, we can construct an extreme case with little deviations from constant returns to scale in both production and education, yet we are still able to generate an indeterminate BGP. This finding echoes what is found in Mino (2001), that a small deviation from constant returns to scale is able to generate indeterminacy in an endogenous growth model, where all externalities are positive and sector-specific, and social returns to scale are constant. We have found the same result in a different model with externalities of average human capital in both production and education sectors and with decreasing social returns to scale in education.

For instance, if $\rho = 0.05$, $A = 1$, $B = 0.03$, $\gamma = 0.01$, $\beta = 0.33$, $\eta = 0.68$, $\alpha = 0.32$ and $g^* = 0.03$, then both production and education sectors demonstrate less than 1% human capital externalities (positive for production but negative for education). However, these parameters can be shown to satisfy conditions in Proposition 7 and therefore can engender indeterminacy, as long as σ is small enough. Though the externalities of human capital are tiny in both production and education sectors, the externalities are essential for indeterminacy: Without these externalities, indeterminacy could never emerge no matter how elastic intertemporal substitution is (not even under linear utility).

4.3 *The case* $\alpha > 0$ *and* $\eta < 1 - \alpha$

Due to the analytical complexity in this case, we present the results with the help of numerical simulations. Of interest are the effects of the human capital spillover in production through γ and the output elasticity of human capital in education through η on the stability. To focus on them, we fix the values of ρ, σ, A, β, α, and the balanced growth rate g^*. When we fix g^*, we vary the total factor productivity in education, B, so that equation (30) holds. There are two reasons we come up with this approach: First, in this way, we can avoid solving the non-polynomial equation (30) in terms of g and dealing with the possible multiplicity of roots; Second, growth rates are observable, yet not so for the total factor productivity of education. Elastic intertemporal substitution is assumed ($\sigma < 1$), so that there is potential for the emergence of indeterminacy within the region of permissible values of γ and η.

We present the results in a set of figures. Figure 5 suggests that the BGP is determinate for small and intermediate values of γ and η, while indeterminacy emerges when both of them increase to relatively large values. This is consistent with the findings from the two cases in the previous subsections. Comparing the diagrams for two values of α in Figure 6, it suggests that, ceteris paribus, a higher output elasticity of physical capital in education, α, loosens the constraints on both the human capital spillover γ and the output elasticity of human capital in education η. This is consistent with the intuition that the complementarity between physical and human capital in education enhances the effectiveness of the intersectoral re-allocation, for constructing alternative converging paths to the BGP. Moreover, Figure 7 reconfirms the conventional finding that more elastic intertemporal substitution enhances the emergence of indeterminacy.

4.4 *Different stability properties for multiple BGPs*

In the situation with multiple BGPs, we cannot rule out the possibility that they do not share the same stability properties. It is possible that the low-growth BGP is determinate, while the high-growth one is indeterminate. However, is the converse possible as well?

Proposition 8. *Given that $g_{low} < g_{high}$ are two possible balanced growth rates, if the BGP associated with g_{high} is determinate, then the other with g_{low} is also determinate.*

See the Appendix for the proof. Therefore, when there are multiple BGPs, it is possible to have an indeterminate high-growth BGP and a determinate low-growth BGP. However, it is impossible to have an opposite pair of the BGPs. One numerical example is (shown in Figure 3): $g_{low} = 1.95\%$ and $g_{high} = 7.01\%$ are two possible balanced growth rates for the economy under $\rho = 0.05$, $\sigma = 0.3$, $A = 1$, $B = 0.0435$, $\gamma = 0.6$, $\beta = 0.33$, $\eta = 0.8$ and $\alpha = 0.08$. It can be calculated that the low-growth BGP is determinate while the high-growth BGP is indeterminate. The existence of such differing stability features across BGPs is not specific to this case: It can be calculated that both the examples for multiple BGPs under $\alpha = 0$ (shown in Figure 1) and under $\alpha + \eta = 1$ (shown in Figure 2) also generate a determinate low-growth BGP and an indeterminate high-growth BGP.

If we interpret determinacy as relatively stable and indeterminacy as more fluctuating driven by self-fulfilling expectations, then the economy may experience slower yet more stable growth or faster yet less stable growth. This is consistent with the growth experiences of Philippine and South Korea, two countries that were similar in many respects in 1960: South Korea grew faster with higher volatility than Philippine from 1960 to 2012.

The source for multiple BGPs in the present model hinges on decreasing returns to scale in education and increasing social returns in production, given elastic enough intertemporal substitution. The low (high) growth BGP accompanies greater (smaller) shares of human and physical capital used for production and consequently a higher (lower) ratio of physical capital to an adjusted indicator for human capital. Therefore, the low-growth BGP benefits (suffers) less from the increasing (decreasing) returns to scale in production (education) from human capital externalities than the high-growth BGP. Given that the increasing returns to scale in production via human capital externalities are the source for indeterminacy on BGPs, it is possible to have a pair of a determinate low-growth BGP and an indeterminate high-growth BGP, whereas it is impossible to have a pair of the opposite. Absent these additional factors in the education sector, the BGP would be unique as in the literature mentioned above.

5 Conclusion

In this paper, we have studied the existence, multiplicity, and indeterminacy of BGPs in an extended version of the Lucas model, by taking into consideration several plausible factors in education: physical capital, human capital externalities and decreasing returns to scale. These extensions lead to several contributions. First, despite decreasing private or social returns to scale in education, long run endogenous growth can still be sustainable in the presence of increasing returns to scale in production via human capital externalities. Also, indeterminacy could emerge for weaker human capital externalities than found in the literature, which eases the concern about whether such strong human capital externalities for indeterminacy as in the literature are plausible. Moreover, multiple BGPs arise under decreasing returns to scale in education and increasing returns to scale in production, given elastic enough intertemporal substitution. The high-growth BGP may be indeterminate, whereas the low-growth BGP is determinate, but not vice versa.

The results may help the understanding of diverse development experiences of nations not only in the short run but also in the long run for empirically plausible education technologies. The results may also help explain diverse growth performance in the same country at different long periods in the absence of fundamental changes. One common requirement of the results is human capital externalities with which agents in a large population fail to coordinate for mutually beneficial actions. One implication of the results is to enhance social coordination for more investment in human capital for a potential gain in efficiency even though the returns to scale are decreasing in education. In the past several decades, emerging economies like Philippine have generally increased public spending on education relative to GDP (World Bank Data). In contrast, Japan has reduced the share of GDP for education substantially from 5.2% (a typical level in developed countries) in 1980 to 3.4% in 2008 (a familiar level in developing countries). Combining such data with the mechanics of the present model, the switch of growth regimes among these countries becomes less mysterious.

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Appendix

Proof of Lemma 1

Proof. Equations (38) and (39) lead to

$$\begin{aligned}
\det(J) &= J_{13}J_{21}J_{32} - J_{23}J_{31} + J_{21}J_{33} - J_{11}J_{23}J_{32} \\
&= \frac{c^*}{k^*}Q^{-1}\left\{-\frac{\alpha\eta}{1-u^*}\frac{1-\beta}{1-\beta+\gamma}g^{*2} + \left[-(1-\beta)(\eta+\alpha) + (\eta+\alpha)\frac{1-\beta}{\sigma}\right.\right. \\
&\quad \left.\left.\times\frac{\gamma}{1-\beta+\gamma} + \frac{\eta}{1-u^*}\frac{1-\beta}{\sigma}\frac{1-\beta}{1-\beta+\gamma}(\eta+\alpha-1)\right] g^*(\sigma g^* + \rho)\right\}. \qquad (42)
\end{aligned}$$

Substituting u^* with the expression in (31), we simplify (42) as

$$\begin{aligned}
\det(J)Q\frac{k^*}{c^*} &= -\left(\frac{1-\alpha-\eta}{\sigma}\right)(1-\beta+\gamma)\left(\frac{\sigma g^*+\rho}{g^*}\right)^2 + \\
&\quad \left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right)\right]\frac{\sigma g^*+\rho}{g^*} + \frac{\alpha\gamma}{1-\beta}, \qquad (43)
\end{aligned}$$

which determines the sign of $\det(J)$ since $Q > 0$ as shown below (21). $\square$

Proof of Proposition 2

Proof. First, $\det(J) < 0$ if $\theta \in \Theta_1$ by Lemma 1, and we will show that conditions (i) and (ii) are sufficient (but not necessary) for a negative determinant at the steady state. From (43), we denote $z = (\sigma g + \rho)/g$ and define

$$f(z) = -\left(\frac{1-\alpha-\eta}{\sigma}\right)(1-\beta+\gamma)z^2 + \left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right)\right]z + \frac{\alpha\gamma}{1-\beta}.$$

Then, the sign of $\det(J)$ is determined by that of $f(z^*)$. If $z = 1$, then

$$\begin{aligned}
f(1) &= -\left(\frac{1-\alpha-\eta}{\sigma}\right)(1-\beta+\gamma) + \frac{\gamma}{\sigma} - (1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right) + \frac{\alpha\gamma}{1-\beta} \\
&= \frac{1-\sigma}{\sigma}(1-\beta+\gamma)(\alpha+\eta) - \frac{1-\beta}{\sigma} - \alpha \\
&\leq 0, \text{ if } \sigma \geq 1 \text{ or else if } \eta \leq \frac{1-\beta+\alpha\sigma}{(1-\sigma)(1-\beta+\gamma)} - \alpha.
\end{aligned}$$

Since the graph of f is a parabola that opens downward and $f(0) = \alpha\gamma/(1-\beta) \geq 0$, $f(1) \leq 0$ is sufficient for $f(z^*) < 0$, as the transversality condition $\sigma g^* + \rho > g^*$ implies that $z^* > 1$.

Second, as the determinant of a matrix equals the product of its eigenvalues, it implies that there are only two possibilities regarding the signs of the eigenvalues: a) all three eigenvalues are negative; and b) only one of them is negative. We will prove that only the latter is possible, by considering two cases: In the first case, the trace of the Jacobian matrix is non-negative. Since the trace of a matrix equals the sum of its eigenvalues, it rules out possibility a) that all three eigenvalues are negative. Therefore, one and only one of the three eigenvalues is negative and the BGP is determinate.

In the second case, the trace of the Jacobian matrix is negative. To prove the determinacy in this case, we make use of Theorem 1 in Jess and Benhabib (1994): "The number of eigenvalues with positive real parts is equal to the number of variations of sign in the scheme

$$-1, \quad \mathrm{tr}(J), \quad -\mathrm{B}(J)+\frac{\det(J)}{\mathrm{tr}(J)}, \quad \det(J)."$$

Since the second and the last terms in the scheme above are negative, we have to show that the third term is positive, to prove that two of the eigenvalues are positive. Note in this case that $\det(J)/\mathrm{tr}(J)$ is positive. Then, it suffices to show that $B(J)$ is non-positive on the BGP with the omission of asterisks for easy of notations:

$$\begin{aligned}
\mathrm{B}(J) &= J_{21}-J_{23}J_{32}+J_{11}J_{33}-J_{13}J_{31} \\
&= (\alpha+T_1+T_2)g^2+\left(\frac{1-\beta}{\sigma}-\frac{\alpha\nu}{\beta}+T_3+T_4+T_5+T_6+T_7\right. \\
&\quad \left.+T_8+T_9\right)g(\sigma g+\rho)+\left(T_{10}+T_{11}-\frac{\nu}{\beta}\frac{1-\beta}{\sigma}\right)(\sigma g+\rho)^2,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= Q^{-1}\left(\alpha+\frac{\alpha\gamma}{1-\beta}-\beta+\gamma\right)\frac{1-\beta}{1-\beta+\gamma}\left(\eta+\frac{\alpha\nu}{u}\right), \\
T_2 &= Q^{-1}\frac{\eta}{1-u}\frac{1-\beta}{1-\beta+\gamma}\left[\frac{\alpha(\nu-u)}{u}+\eta-1\right], \\
T_3 &= (\beta-\alpha)\frac{1}{u}Q^{-1}\left[-\frac{1-\beta}{\sigma}(\nu-u)+\frac{\nu u}{\beta}\left(\eta+\frac{\alpha\nu}{u}\right)\right],
\end{aligned}$$

$$
\begin{aligned}
T_4 &= (1-u)Q^{-1}\frac{1+\alpha}{\beta}\frac{(\alpha-\beta)\nu[(1-\beta)u+\beta D\nu]+\beta(1-\beta)(u-D\nu)}{u^2},\\
T_5 &= -Q^{-1}\left(\eta+\frac{\alpha\nu}{u}\right)[1-\beta-(\alpha-\beta)\nu],\\
T_6 &= (1-u)Q^{-1}(\beta-\alpha)\frac{\nu}{u}\left(\frac{1-\beta}{\beta}+\frac{D\nu}{u}\right),\ T_7=-T_1,\\
T_8 &= Q^{-1}\frac{1-\beta}{1-\beta+\gamma}\eta\frac{\nu}{1-u},\\
T_9 &= -Q^{-1}\left(\eta+\frac{\alpha\nu}{u}\right)\frac{1-\beta}{1-\beta+\gamma}\frac{\eta}{1-u},\\
T_{10} &= -\frac{(1-\beta)}{\beta}\frac{\nu(1-u)}{u}Q^{-1},\ \text{and}\\
T_{11} &= \frac{(\beta-\alpha)}{\beta}\frac{\nu(\nu-u)}{u}\frac{1-\beta}{\sigma}Q^{-1}.
\end{aligned}
$$

Recall that $Q>0$ below (21). The negative sign of the trace in this case means

$$
\mathrm{tr}(J)=J_{11}+J_{33}=-(1+\alpha+T_1+T_2)g+\left(\nu-\frac{T_4}{1+\alpha}\right)(\sigma g+\rho)<0,
$$

where $\nu-T_4/(1+\alpha)>1$ as shown below:

$$
\begin{aligned}
-\frac{T_4}{1+\alpha} &= -(1-u)Q^{-1}\frac{(\alpha-\beta)\nu[(1-\beta)u+\beta D\nu]+\beta(1-\beta)(u-D\nu)}{\beta u^2}\\
&= -Q^{-1}\frac{\nu(1-u)}{\beta u}\{(\alpha-\beta)+\beta(1-D)[(\beta-\alpha)\nu+1-\beta]\}\\
&\geq \frac{D+(1-D)u}{(\alpha-\beta)(1-D)(1-u)}\frac{\nu(1-u)}{\beta u}\left\{(\alpha-\beta)+\beta\frac{\beta-\alpha}{\beta(1-\alpha)}[(\beta-\alpha)\nu+1-\beta]\right\}\\
&= \frac{\beta-\alpha}{\beta(1-\alpha)(1-D)}(1-\nu)>1-\nu,
\end{aligned}
$$

since $\beta>\alpha$ under Assumption 2 and $0<D=\alpha(1-\beta)/(\beta\eta)<1$ under Assumption 3. Then, we get $\alpha+T_1+T_2>0$ and thus,

$$
\begin{aligned}
\mathrm{B}(J) &= (\alpha+T_1+T_2)\,g^2+\left(\frac{1-\beta}{\sigma}-\frac{\alpha\nu}{\beta}+T_3+T_4+T_5+T_6+T_7\right.\\
&\qquad \left.+T_8+T_9\right)g(\sigma g+\rho)+\left(T_{10}+T_{11}-\frac{\nu}{\beta}\frac{1-\beta}{\sigma}\right)(\sigma g+\rho)^2\\
&\leq \left(\alpha+T_1+T_2+\frac{1-\beta}{\sigma}-\frac{\alpha\nu}{\beta}+T_3+T_4+T_5+T_6+T_7\right.
\end{aligned}
$$

$$+ T_8 + T_9\Big) g(\sigma g + \rho) + \left(T_{10} + T_{11} - \frac{\nu}{\beta}\frac{1-\beta}{\sigma}\right)(\sigma g + \rho)^2$$

$$= \left(\underbrace{\alpha - \frac{\alpha\nu}{\beta}}_{T_a} + \underbrace{\frac{1-\beta}{\sigma} + T_3 + T_5}_{T_b} + \underbrace{T_4 + T_6}_{T_c} + \underbrace{T_2 + T_8 + T_9}_{T_d}\right) g(\sigma g + \rho)$$

$$+ \left(T_{10} + \underbrace{T_{11} - \frac{\nu}{\beta}\frac{1-\beta}{\sigma}}_{T_e}\right)(\sigma g + \rho)^2.$$

We analyze the signs of those terms. First, $T_a \equiv \alpha - \alpha\nu/\beta \leq 0$ as $\nu/\beta \geq 1$ by (32) and the transversality condition. Second, define $T_b \equiv (1-\beta)/\sigma + T_3 + T_5$ and $T_e \equiv T_{11} - \frac{\nu}{\beta}(1-\beta)/\sigma$, and by substitution we get:

$$\begin{aligned}
T_b &= \frac{1-\beta}{\sigma} + (\beta-\alpha)\frac{1}{u}Q^{-1}\left[-\frac{1-\beta}{\sigma}(\nu-u) + \frac{\nu u}{\beta}\left(\eta + \frac{\alpha\nu}{u}\right)\right] \\
&\quad - Q^{-1}\left(\eta + \frac{\alpha\nu}{u}\right)[1-\beta-(\alpha-\beta)\nu] \\
&= \left[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)\right]\frac{1-\beta}{\sigma} - Q^{-1}\left(\eta + \frac{\alpha\nu}{u}\right)(1-\beta)\left(1 - \frac{\beta-\alpha}{\beta}\nu\right), \\
T_e &= -\frac{\nu}{\beta}[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)]\frac{1-\beta}{\sigma}.
\end{aligned}$$

It can be shown that $T_e \leq 0$ by (19) and the definition of Q, and $\nu/\beta \geq 1$ by (32) and the transversality condition. Then, we have

$$\begin{aligned}
&T_b g(\sigma g + \rho) + T_e(\sigma g + \rho)^2 \leq (T_b + T_e) g(\sigma g + \rho) \\
&= \left\{\left[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)\right]\frac{1-\beta}{\sigma} - \frac{\nu}{\beta}\left[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)\right]\frac{1-\beta}{\sigma}\right. \\
&\quad \left. - Q^{-1}\left(\eta + \frac{\alpha\nu}{u}\right)(1-\beta)\left(1 - \frac{\beta-\alpha}{\beta}\nu\right)\right\} g(\sigma g + \rho) \\
&\leq \left\{\left[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)\right]\frac{1-\beta}{\sigma} - \left[1 - (\beta-\alpha)\frac{1}{u}Q^{-1}(\nu-u)\right]\frac{1-\beta}{\sigma}\right. \\
&\quad \left. - Q^{-1}\left(\eta + \frac{\alpha\nu}{u}\right)(1-\beta)\left(1 - \frac{\beta-\alpha}{\beta}\nu\right)\right\} g(\sigma g + \rho) \\
&= -Q^{-1}\left(\eta + \frac{\alpha\nu}{u}\right)(1-\beta)\left(1 - \frac{\beta-\alpha}{\beta}\nu\right) g(\sigma g + \rho) \leq 0.
\end{aligned}$$

Next, we can also show that $T_c g(\sigma g + \rho) + T_{10}(\sigma g + \rho)^2 \leq 0$ as follows:

$$
\begin{aligned}
& T_c g(\sigma g + \rho) + T_{10}(\sigma g + \rho)^2 \\
&= \left\{(1-u)Q^{-1}\frac{1+\alpha}{\beta}\frac{(\alpha-\beta)\nu[(1-\beta)u + \beta D\nu] + \beta(1-\beta)(u - D\nu)}{u^2}\right. \\
&\left. +(1-u)Q^{-1}(\beta-\alpha)\frac{\nu}{u}\left(\frac{1-\beta}{\beta} + \frac{D\nu}{u}\right)\right\} g(\sigma g + \rho) - \frac{(1-\beta)}{\beta}\frac{\nu(1-u)}{u}Q^{-1}(\sigma g + \rho)^2 \\
&= \left[-\alpha(\beta-\alpha)\frac{\nu(1-u)}{u}\left(\frac{1-\beta}{\beta} + \frac{D\nu}{u}\right)Q^{-1} + (1+\alpha)(1-\beta)\frac{(1-D)\nu(1-u)}{u}\right. \\
&\left. \cdot Q^{-1}\right] g(\sigma g + \rho) - \frac{(1-\beta)}{\beta}\frac{\nu(1-u)}{u}Q^{-1}(\sigma g + \rho)^2 \\
&< \left[-\alpha(\beta-\alpha)\frac{\nu(1-u)}{u}(\frac{1-\beta}{\beta} + \frac{D\nu}{u})Q^{-1} + (1+\alpha)(1-\beta)\frac{(1-D)\nu(1-u)}{u}Q^{-1}\right. \\
&\left. -\frac{(1-\beta)}{\beta}\frac{\nu(1-u)}{u}Q^{-1}\right] g(\sigma g + \rho) \\
&< \left[-\alpha(\beta-\alpha)\frac{\nu(1-u)}{u}(\frac{1-\beta}{\beta} + D) + \alpha(1-\beta)\frac{(1-D)\nu(1-u)}{u}\right] Q^{-1} g(\sigma g + \rho) \leq 0,
\end{aligned}
$$

since the terms in the brackets

$$
\begin{aligned}
[\cdots] &= \alpha\frac{\nu(1-u)}{u}\left[-(\beta-\alpha)\left(\frac{1-\beta}{\beta} + D\right) + (1-\beta)(1-D)\right] \\
&= \alpha D\frac{\nu(1-u)}{u}[\alpha + \beta\eta - 1] \leq 0.
\end{aligned}
$$

Finally, we show that $T_d \leq 0$:

$$
T_d \equiv T_2 + T_8 + T_9 = Q^{-1}\frac{\eta}{1-u}\frac{1-\beta}{1-\beta+\gamma}[-\alpha - 1 + \nu] \leq 0,
$$

since $Q > 0$ as shown below (21). Therefore, we get that $\mathrm{B}(J) \leq 0$ as

$$
\begin{aligned}
\mathrm{B}(J) &= (T_a + T_b + T_c + T_d)g(\sigma g + \rho) + (T_{10} + T_e)(\sigma g + \rho)^2 \\
&= T_a g(\sigma g + \rho) + \left[T_b g(\sigma g + \rho) + T_e(\sigma g + \rho)^2\right] \\
&\quad + \left[T_c g(\sigma g + \rho) + T_{10}(\sigma g + \rho)^2\right] + T_d g(\sigma g + \rho) \\
&\leq 0 + 0 + 0 + 0 = 0.
\end{aligned}
$$

□

Proof of Proposition 3

Proof. First, $\det(J) > 0$ if $\theta \in \Theta_2$ by the definition of Θ_2. As the determinant equals the product of the three eigenvalues, this implies that there are only two possibilities: a) all three eigenvalues are positive; and b) two of them are negative and the other one is positive. If we denote the two subsets of Θ_2 that generate these two cases by Θ_2^U for case a) and Θ_2^I for case b), respectively, then the steady state is unstable if $\theta \in \Theta_2^U$ and the steady state is indeterminate if $\theta \in \Theta_2^I$. □

Proof of Proposition 4

Proof. According to Proposition 2, the steady state is determinate if and only if $\theta \in \Theta_1$. The restriction $\alpha = 0$ in this case implies:

$$\begin{aligned}\Theta_1 &\equiv \left\{\theta \in \Theta \mid -\frac{1-\alpha-\eta}{\sigma}(1-\beta+\gamma)\left(\frac{\sigma g^* + \rho}{g^*}\right)^2 + \left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\right.\right.\\ &\qquad \left.\left.\left(\frac{\alpha}{1-\beta} + \eta + \alpha\right)\right]\frac{\sigma g^* + \rho}{g^*} + \frac{\alpha\gamma}{1-\beta} < 0\right\}\\ &= \left\{\theta \in \Theta \mid -\frac{1-\eta}{\sigma}(1-\beta+\gamma)\frac{\sigma g^* + \rho}{g^*} + \frac{\gamma}{\sigma} - (1-\beta+\gamma)\eta < 0\right\}\\ &= \left\{\theta \in \Theta \mid \eta < 1 - \frac{g^*}{\rho}\left(\frac{\gamma}{1-\beta+\gamma} - \sigma\right)\right\}.\end{aligned}$$

□

Proof of Proposition 5

Proof. First of all, we show that $\det(J)$ is positive if and only if condition (i) is true. As shown in the proof of Proposition 4, given $\alpha = 0$, $\det(J)$ shares the same sign as

$$-\frac{1-\eta}{\sigma}(1-\beta+\gamma)\frac{\sigma g^* + \rho}{g^*} + \frac{\gamma}{\sigma} - (1-\beta+\gamma)\eta,$$

which is positive if and only if condition (i) is met.

Next, since a positive $\det(J)$ is necessary for an indeterminate steady state, condition (i) is actually a necessary condition for the indeterminacy. Thus, there are only two possibilities regarding the signs of the three eigenvalues: a) all three are positive; and b) one is positive and two are negative. We prove that only the latter is possible, given conditions (ii) in the proposition, by considering two cases: In the first case, the trace of the Jacobian matrix is non-positive. This rules out the possibility that all three eigenvalues are positive. Therefore, one and only one of the eigenvalues is positive in this case and the BGP is indeterminate.

In the second case, the trace of the Jacobian matrix is positive. Again to prove the indeterminacy in this case, we make use of Theorem 1 in Benhabib and Perli (1994). It suffices to prove that $B(J) \leq 0$: We can still use the expression of $\mathrm{B}(J)$ as in the proof of Proposition 2, except that $\alpha = 0$ here:

$$\begin{aligned}\mathrm{B}(J) \quad &= \quad (\alpha + T_1 + T_2)\, g^2 + \left(\frac{1-\beta}{\sigma} - \frac{\alpha\nu}{\beta} + T_3 + T_4 + T_5 + T_6 + T_7 \right. \\ &\qquad \left. + T_8 + T_9\right) g(\sigma g + \rho) + \left(T_{10} + T_{11} - \frac{\nu}{\beta}\frac{1-\beta}{\sigma}\right)(\sigma g + \rho)^2.\end{aligned}$$

We show that $\alpha + T_1 + T_2 \geq 0$ as follows:

$$\begin{aligned}\alpha + T_1 + T_2 \quad &= \quad 0 + Q^{-1}(\gamma - \beta)\frac{1-\beta}{1-\beta+\gamma}\eta - Q^{-1}\frac{\eta}{1-u^*}\frac{1-\beta}{1-\beta+\gamma}(1-\eta) \\ &= \quad Q^{-1}\frac{1-\beta}{1-\beta+\gamma}\eta(\gamma - \beta - \frac{1-\eta}{1-u^*}) \\ &\geq \quad 0 \text{ if } \eta \geq 1 - (\gamma-\beta)(1-u^*).\end{aligned}$$

We can substitute u^* with the expression in (31) and write this constraint as condition (ii) in the proposition. The rest of the proof is the same as that of Proposition 2. □

Proof of Proposition 6

Proof. According to Proposition 2, the steady state is determinate if and only if $\theta \in \Theta_1$. The restriction $\eta = 1 - \alpha$ in this case implies:

$$\Theta_1 \quad \equiv \quad \left\{\theta \in \Theta \mid -\frac{1-\alpha-\eta}{\sigma}(1-\beta+\gamma)\left(\frac{\sigma g^* + \rho}{g^*}\right)^2 + \left[\frac{\gamma}{\sigma} - (1-\beta+\gamma)\right.\right.$$

$$\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right)\Bigg]\frac{\sigma g^*+\rho}{g^*}+\frac{\alpha\gamma}{1-\beta}<0\Bigg\}$$

$$= \left\{\theta\in\Theta \mid \left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+1\right)\right]\frac{\sigma g^*+\rho}{g^*}+\frac{\alpha\gamma}{1-\beta}<0\right\}$$

$$= \left\{\theta\in\Theta \mid \eta=1-\alpha<1-\frac{(1-\beta)(\sigma+\rho/g^*)[\gamma/\sigma-(1-\beta+\gamma)]}{(1-\beta+\gamma)(\sigma+\rho/g^*)-\gamma}\right\}.$$

□

Proof of Proposition 7

Proof. First of all, we show that $\det(J)$ is positive if and only if condition (i) is true. As shown in the proof of Proposition 6, given $\eta=1-\alpha$, $\det(J)$ shares the same sign as

$$\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+1\right)\right]\frac{\sigma g^*+\rho}{g^*}+\frac{\alpha\gamma}{1-\beta},$$

which is positive if and only if condition (i) is met.

The rest of the proof can follow that of Proposition 5, except the part for $\mathrm{B}(J)\leq 0$. We can still use the expression of $\mathrm{B}(J)$ as in the proof of Proposition 2, except that $\eta=1-\alpha$ here:

$$\begin{aligned}\mathrm{B}(J) &= (\alpha+T_1+T_2)\,g^2+\left(\frac{1-\beta}{\sigma}-\frac{\alpha\nu}{\beta}+T_3+T_4+T_5+T_6+T_7\right.\\ &\qquad \left.+T_8+T_9\right)g(\sigma g+\rho)+\left(T_{10}+T_{11}-\frac{\nu}{\beta}\frac{1-\beta}{\sigma}\right)(\sigma g+\rho)^2.\end{aligned}$$

Condition (ii) in the proposition ensures that $T_1\geq 0$. There are two cases regarding the sign of T_2: If $T_2\geq 0$, then the derivation in the proof of Proposition 2 can be followed for $\mathrm{B}(J)\leq 0$; and if $T_2<0$, then the same derivation can still be followed, except the part on T_8 and T_9. Here we prove that $T_8+T_9\leq 0$ when $\eta=1-\alpha$:

$$\begin{aligned}T_8+T_9 &= Q^{-1}\frac{1-\beta}{1-\beta+\gamma}\eta\frac{\nu}{1-u}-Q^{-1}\left(\eta+\frac{\alpha\nu}{u}\right)\frac{1-\beta}{1-\beta+\gamma}\frac{\eta}{1-u}\\ &= Q^{-1}\frac{1-\beta}{1-\beta+\gamma}\frac{\eta}{1-u}\left(\nu-\eta-\frac{\alpha\nu}{u}\right)\\ &= Q^{-1}\frac{1-\beta}{1-\beta+\gamma}\frac{\eta}{1-u}\left(\nu-1+\alpha-\frac{\alpha\nu}{u}\right)\end{aligned}$$

The last step made use of $\nu>u$, which can be derived from equation (18). □

Proof of Proposition 8

Proof. If $\alpha+\eta=1$, from Proposition 6, the high-growth steady state being determinate implies

$$\eta < 1 - \frac{(1-\beta)(\sigma+\rho/g_{high})[\gamma/\sigma-(1-\beta+\gamma)]}{(1-\beta+\gamma)(\sigma+\rho/g_{high})-\gamma}.$$

Given $\sigma \leq \gamma/(1-\beta+\gamma)$ for multiple steady states, we have

$$\begin{aligned}
1 \quad - \quad & \frac{(1-\beta)\left(\sigma+\frac{\rho}{g_{high}}\right)\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\right]}{(1-\beta+\gamma)\left(\sigma+\frac{\rho}{g_{high}}\right)-\gamma} \\
= \quad & 1 - \frac{(1-\beta)\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\right]}{(1-\beta+\gamma)-\gamma/\left(\sigma+\frac{\rho}{g_{high}}\right)} \\
\leq \quad & 1 - \frac{(1-\beta)\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\right]}{(1-\beta+\gamma)-\gamma/\left(\sigma+\frac{\rho}{g_{low}}\right)} \\
= \quad & 1 - \frac{(1-\beta)\left(\sigma+\frac{\rho}{g_{low}}\right)\left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\right]}{(1-\beta+\gamma)\left(\sigma+\frac{\rho}{g_{low}}\right)-\gamma}.
\end{aligned}$$

Therefore, the low-growth steady state also satisfies the necessary and sufficient condition in Proposition 6 and is also determinate.

If $\alpha+\eta<1$, we denote $x_i=(\sigma g_i+\rho)/g_i$, where i refers to either 'high' or 'low', and define

$$f(x_i) = -\frac{1-\alpha-\eta}{\sigma}(1-\beta+\gamma)x_i^2 + \left[\frac{\gamma}{\sigma}-(1-\beta+\gamma)\left(\frac{\alpha}{1-\beta}+\eta+\alpha\right)\right]x_i + \frac{\alpha\gamma}{1-\beta}.$$

From Proposition 2 and the definition of Θ_1, the high-growth steady state being determinate implies $f(x_{high})<0$. We prove $f(x_{low})<0$, given $f(x_{high})<0$. Note that $f(x)$ is a quadratic function and its graph is a parabola that opens downward, as $-(1-\alpha-\eta)(1-\beta+\gamma)/\sigma<0$. Since $f(0)=\alpha\gamma/(1-\beta)\geq 0 > f(x_{high})$, the axis of symmetry should be to the left of x_{high}, implying that f declines on $[x_{high},\ +\infty)$. Also, $g_{low}<g_{high}$ means $x_{low}>x_{high}$ since

$$x_{low} = \frac{\sigma g_{low}+\rho}{g_{low}} = \sigma+\frac{\rho}{g_{low}} > \sigma+\frac{\rho}{g_{high}} = \frac{\sigma g_{high}+\rho}{g_{high}} = x_{high},$$

thus $f(x_{low})<f(x_{high})<0$. Therefore, the low-growth steady state also satisfies the necessary and sufficient condition in Proposition 2 and is also determinate. $\square$

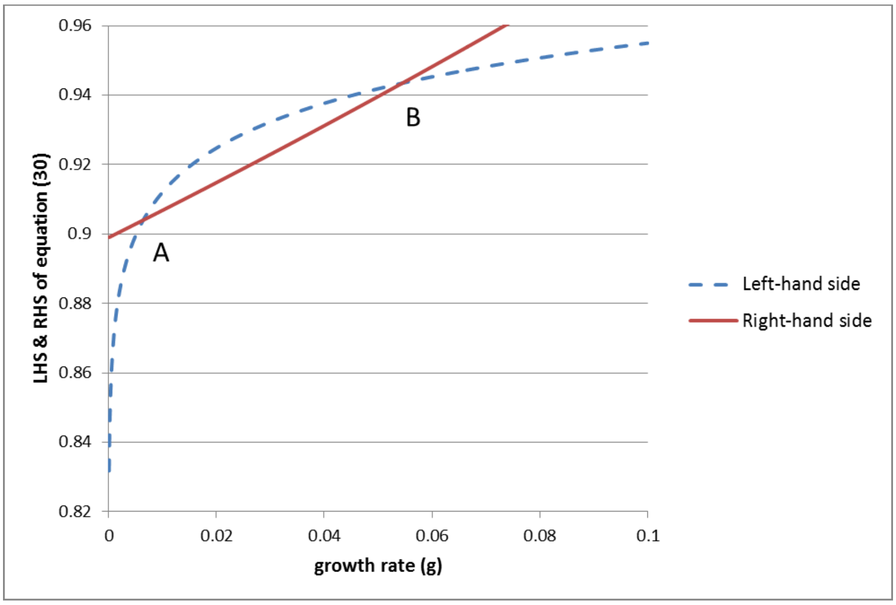


Figure 1: Multiple steady states under $\alpha = 0$

We calculate the left-hand and right-hand sides of equation (30), under $\rho = 0.07$, $\sigma = 0.34$, $A = 1.1$, $B = 0.067$, $\gamma = 0.45$, $\beta = 0.33$, $\eta = 0.98$ and $\alpha = 0$, from which we can find that there are two roots for the equation, between 0 and 0.1.

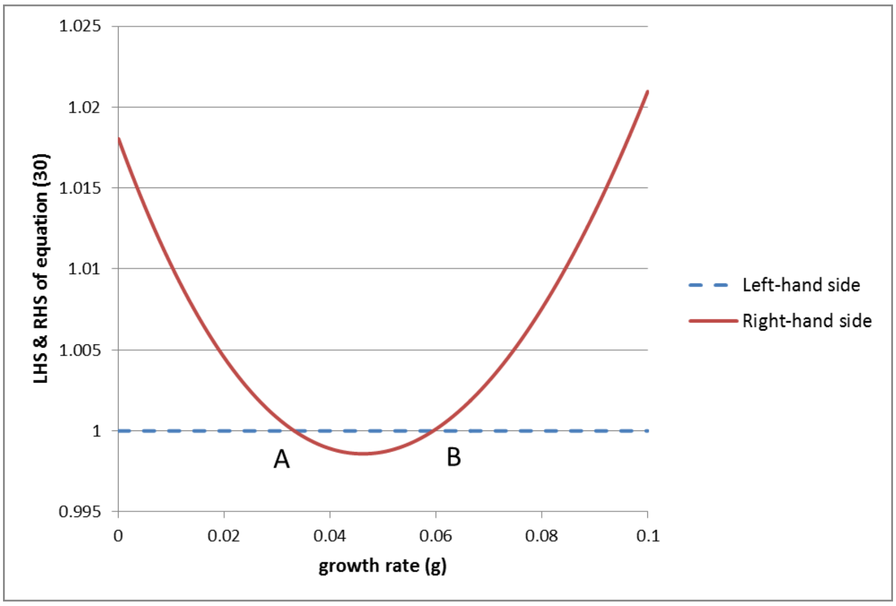


Figure 2: Multiple steady states under $\alpha + \eta = 1$

We calculate the left-hand and right-hand sides of equation (30), under $\rho = 0.045$, $\sigma = 0.28$, $A = 1.433$, $B = 0.022$, $\gamma = 0.4$, $\beta = 0.33$, $\eta = 0.68$ and $\alpha = 0.32$, from which we can find that there are two roots for the equation, between 0 and 0.1.

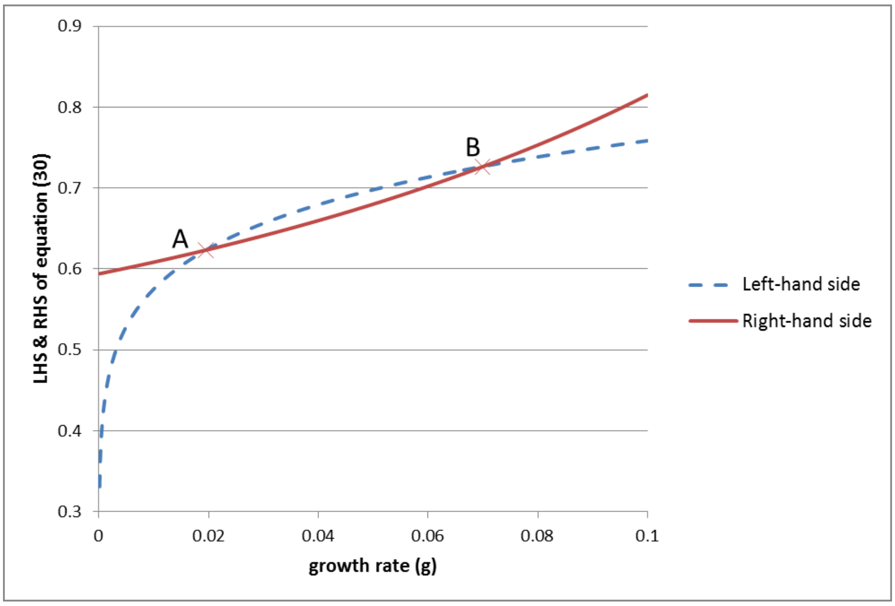


Figure 3: Multiple steady states under $\alpha > 0$ and $\alpha + \eta < 1$
We calculate the left-hand and right-hand sides of equation (30), under $\rho = 0.05$, $\sigma = 0.3$, $A = 1$, $B = 0.0435$, $\gamma = 0.6$, $\beta = 0.33$, $\eta = 0.8$ and $\alpha = 0.08$, from which we can find that there are two roots for the equation, between 0 and 0.1.

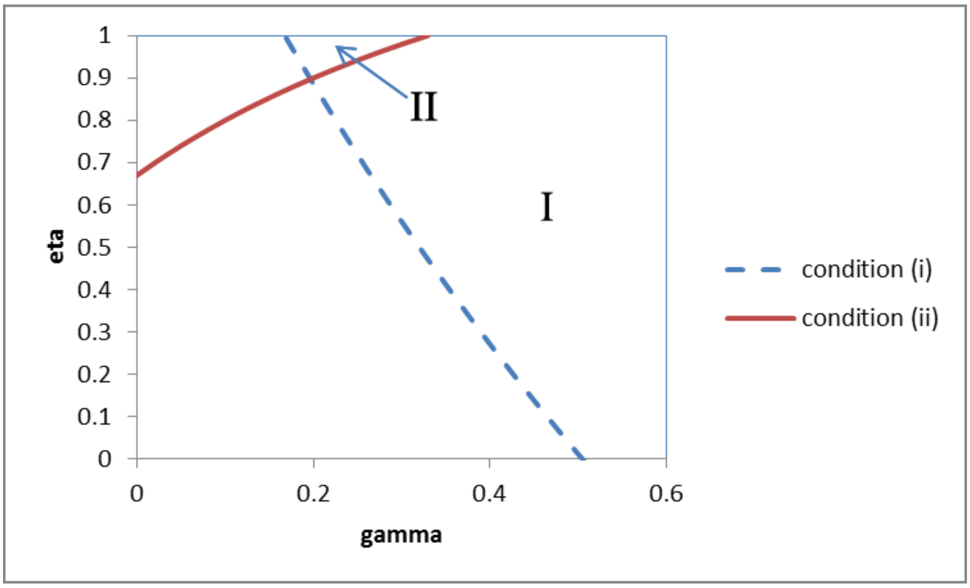


Figure 4: Illustration of conditions (i) and (ii) in Proposition 7
We denote the region to the right of both the solid and the dash lines as region I, and the region to the right of the dash line but to the left of the solid line as region II. Then region I demonstrates indeterminacy, while region II generates either an indeterminate or an unstable BGP.

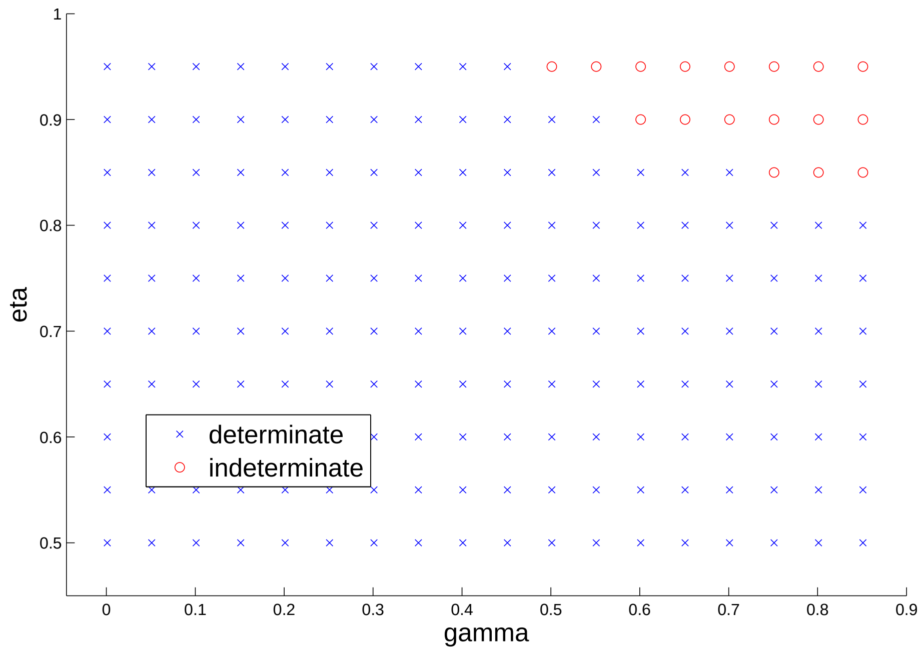
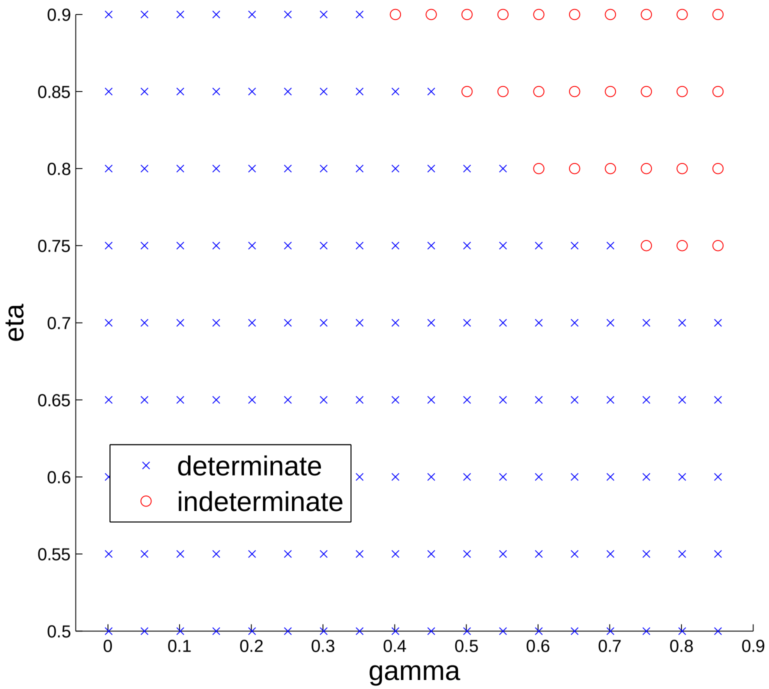
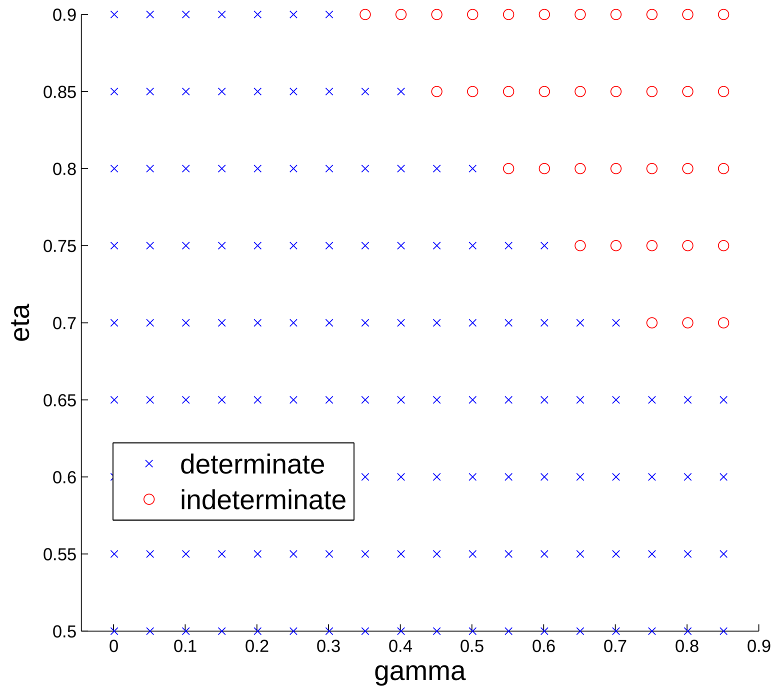


Figure 5: The effect of γ and η on the local stability of the steady state ($\rho = 0.05,\ \sigma = 0.4,\ A = 1,\ \beta = 0.33,\ \alpha = 0.05$ and $g^* = 0.05$)

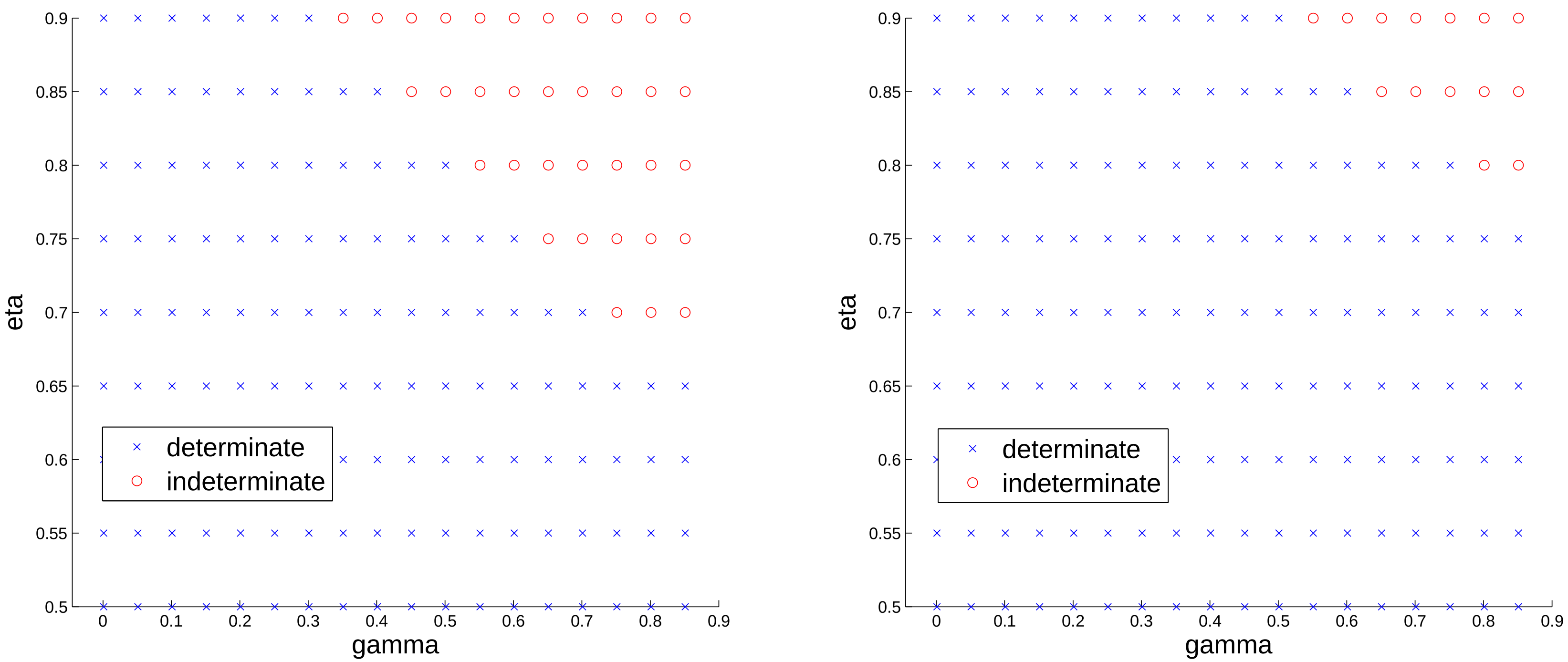


(a) $\alpha = 0.05$



(b) $\alpha = 0.1$

Figure 6: A comparison between two values of α ($\rho = 0.05,\ \sigma = 0.3,\ A = 1,\ \beta = 0.33$ and $g^* = 0.05$)



(a) $\sigma = 0.3$

(b) $\sigma = 0.4$

Figure 7: A comparison between two values of σ ($\rho = 0.05,\ A = 1,\ \beta = 0.33,\ \alpha = 0.1$ and $g^* = 0.05$)